Second Moment Method

Apoorv Vikram Singh

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Q: For which $p = p_n$ does $G(n,p)$ contain a triangle w.p. $1 - o(1)$?

Ask the counter:

When does it NOT contain a triangle w.p. $1 - o(1)$?

FIRST MOMENT

X: no. of Δ's in $G(n,p)$.

$$\mathbb{E}X = \binom{n}{3} p^{\binom{3}{2}} \asymp n^3p^3$$

Markovs: $\mathbb{P}(X \geq 1) \leq \mathbb{E}X$

$$\therefore \mathbb{P}(X > 0) \leq n^3p^3.$$

$\therefore$ If $np \to 0$ then $G(n,p)$ is Δ free w.p. $1-o(1)$.

- WHAT IF $p \gg \frac{1}{n}$?

$\mathbb{E}X \to \infty$. Then

$\mathbb{P}(X>0) < \infty$.

- We want to show for some p,

$\mathbb{P}(X>0) = 1 - o(1)$.

- $\mathbb{E}X$ can be ∞ but X can be 0 whp.

  - why?

$$X_n = \begin{cases} 0 & \text{w.p. } 1-\frac{1}{n} \\ n^7 & \text{w.p. } \frac{1}{n} \end{cases}$$

NOW, SECOND MOMENT:

Idea: If we can show X is conc. around its mean then we are happy

: Concentration. $\therefore$ Look at higher moments.

* Def (variance):

$$\mathrm{Var}\, X = \mathbb{E}[(X - \mathbb{E}X)^2] = \mathbb{E}X^2 - (\mathbb{E}X)^2$$

$$\mathrm{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}X)(Y - \mathbb{E}Y)]$$

$$= \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y].$$

- σ^2: variance, $\sigma \geq 0$: std. deviation.

* **Thm:** X r.v.: $\mathbb{E}[X] = \mu$, $\operatorname{Var} X = \sigma^2$. For $\lambda > 0$

$$\mathbb{P}(|X-\mu| \geq \lambda\sigma) \leq \frac{1}{\lambda^2}.$$

$$\Rightarrow \quad \mathbb{P}(X=0) \leq \mathbb{P}(|X-\mu| \geq |\mu|)$$

$$\leq \frac{\operatorname{Var} X}{\mu^2}.$$

$$\boxed{\begin{array}{l} \therefore \text{ If } \mathbb{E}X > 0, \quad \operatorname{Var} X = o\big((\mathbb{E}X)^2\big), \quad \text{then } X > 0, \\ \quad \checkmark \quad X \sim \mathbb{E}X \quad \text{w.p. } 1 - o(1). \end{array}}$$

$$\begin{aligned} \mathbb{P}(X>0) &= 1 - \mathbb{P}(X=0) \\ &\geq 1 - \frac{\operatorname{Var} X}{\mu^2} \\ &= 1 - o(1) \end{aligned}$$

$\boxed{\begin{array}{l}\text{Thm: If } \mathbb{E}X > 0, \quad \operatorname{Var} X = o(\mathbb{E}X)^2, \quad \text{then } X > 0, \\ \quad \text{and } X \sim \mathbb{E}X \text{ w.p. } 1 - o(1).\end{array}}$

BACK TO TRIANGLES:

$$X_{ij} = \mathbb{1}\{\text{edge } (i,j) \text{ exists in } G(n,p)\}.$$

$$X_{ijk} := X_{ij} X_{ik} X_{jk}$$

$$\text{No. of } \Delta\text{'s} =: X = \sum_{i<j<k} X_{ij} X_{ik} X_{jk}.$$

We know $\mathbb{E}X \asymp n^3 p^3$

Now compute $\operatorname{Var} X$. [INDEPENDENCE NOT NECESSARY]

If T_1 & T_2 are each 3-vertex subsets, then:

$$\mathrm{Cov}(X_{T_1}, X_{T_2}) = \mathbb{E}[X_{T_1} X_{T_2}] - \mathbb{E}[X_{T_1}]\cdot\mathbb{E}[X_{T_2}]$$

$$= p^{e(T_1 \cup T_2)} - p^{e(T_1)+e(T_2)}$$

$$= \begin{cases} 0, & \text{if } |T_1 \cap T_2| \le 1 \\ p^5 - p^6, & \text{if } |T_1 \cap T_2| = 2 \\ p^3 - p^6, & \text{if } T_1 = T_2 \end{cases}.$$

$X_{ij} \in 0,1$

$X_{ij} = X_{ij}^2$

$$\frac{n^3p^3 + n^4p^5}{n^6p^6} \to 0$$

$n^6p^6 \approx (\mathbb{E}X)^2$

$$\therefore \mathrm{Var}(X) = \sum_{T_1, T_2} \mathrm{cov}(X_{T_1}, X_{T_2}) = \binom{n}{3}(p^3 - p^6) + \binom{n}{2}\binom{n-2}{2}(p^5 - p^6)$$

$$(\log n)^3 + (\log n)^2 \le (\log n)^6 \lesssim n^3p^3 + n^4p^5 = o(n^6p^6)$$

as $np \to \infty$

$\therefore \mathrm{Var}\, X = o(\mathbb{E}X)^2 \Rightarrow X > 0$ whp. ∎

- We say $\frac{1}{n}$ is a threshold for containing a Δ.

  i.e. $p \gg \frac{1}{n}$ then Δ w.p. $1-o(1)$

  $p \ll \frac{1}{n}$ then no Δ w.p. $1-o(1)$

- What if $np \to c$?

  # Δ's in $G(n,p)$ approaches Poisson dist with const. mean.

Thresholds for fixed Subgraphs

Setup : Variance with bdd. dependencies :

- Suppose $X = X_1 + \cdots + X_m$, $X_i = \mathbb{1}(A_i)$ → event.
- $i \sim j$ if $i \neq j$ & (A_i, A_j) are NOT independent.

$$\Delta^* := \max_i \sum_{j : j \sim i} \mathbb{P}(A_j \mid A_i).$$

- Δ^* considers only pair-wise dependencies. For more general thing, we consider LLL.

Using the setup: $X = x_1 + \cdots + x_m$

$$\mathrm{cov}(X_i, X_j) = \mathbb{E}[X_i X_j] - \mathbb{E}[X_i]\,\mathbb{E}[X_j] \leq \mathbb{E}[X_i X_j]$$

$$= \mathbb{P}[A_i A_j]$$

$$= \mathbb{P}(A_i) \cdot \mathbb{P}(A_j \mid A_i).$$

$\therefore$

$$\mathrm{Var}\, X = \sum_{i,j=1}^{m} \mathrm{cov}[X_i, X_j] \leq \sum_{i=1}^{m} \mathbb{P}(A_i) + \sum_{i=1}^{m} \mathbb{P}(A_i) \sum_{j : j \sim i} \mathbb{P}(A_j \mid A_i).$$

$$\leq \mathbb{E}X + (\mathbb{E}X)\,\Delta^*.$$

- Lemma:

If $\mathbb{E}X \to \infty$, $\Delta^* = o(\mathbb{E}X)$, then $X > 0$ & $X \sim \mathbb{E}X$ whp.

* Threshold for containing K_4:

· $\mathbb{E}X = \binom{n}{4} p^{\binom{4}{2}} \asymp n^4 p^6$.

$\Rightarrow$ $p \ll n^{-2/3}$ $\ll n^{-0.6}$

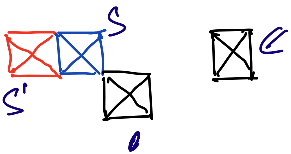
$\Rightarrow$ $X = 0$ whp.

$n^2 p^3 \to \infty$

· Suppose $p \gg n^{-2/3}$.

$A_S :=$ event: S is a 4-clique in $G(n,p)$.

· Fix S. $A_S \sim A_{S'}$ iff $|S \cap S'| \geq 2$



· # S' share exactly 2 vertices: $\binom{4}{2} \cdot \binom{n-4}{2} = O(n^2)$.

& $\mathbb{P}(A_{S'} \mid A_S) = p^5$.

· # S' share exactly 3 vertices: $\binom{4}{3} \cdot \binom{n-4}{1} = O(n)$.

& $\mathbb{P}(A_{S'} \mid A_S) = p^3$.

∴ Summing over all S'

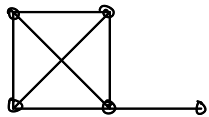
$$\Delta^* = \sum_{S' : |S \cap S'| \in \{2,3\}} \mathbb{P}(A_{S'} \mid A_S) \lesssim n^2 p^5 + n p^3 \ll n^4 p^6 \asymp \mathbb{E}X$$

$$\frac{n^2 p^5 + n p^3}{n^4 p^6} = \frac{1}{n^2 p} + \frac{1}{n^3 p^3}$$

∴ $X > 0$ whp.

Lemma: If $\mathbb{E}X \to \infty$, $\Delta^* = o(\mathbb{E}X)$, then $X > 0$ & $X \sim \mathbb{E}X$ whp.

FIRST MOMENT DOESN'T GIVE RIGHT THRESHOLD

$H =$ 

$$\mathbb{E}X_H \asymp n^5 p^7$$

$\mathbb{E}X = o(1)$, then $X = 0$ whp. i.e. $p \ll n^{-5/7} = n^{-0.71}$

$\therefore$ If $p \gg n^{-0.7}$ then $\mathbb{E}X \to \infty$.

· **But:** $K_4 \subseteq H$. ✓ $p = n^{-2/3} = n^{-0.6}$ is the threshold.

$$\Rightarrow \quad n^{-0.7} \ll p \ll n^{-0.6} \quad \Rightarrow \quad X = 0 \text{ w.h.p.}$$

The right threshold is $p = n^{-2/3}$.

· We need to look at the "densest" subgraph of H.

* Def: edge-vertex ratio of graph H:

$$\rho(H) := \frac{e_H}{v_H}$$

max edge-vertex ratio of subgraph of H

$$m(H) := \max_{H' \subseteq H} \rho(H').$$

* Thm: (Bollobás '1981)

Fix a graph H. Then $p = n^{-1/m(H)}$ is a threshold for containing H as a subgraph.

PROBLEMS:

Q1. *Isolated vertices.* Let $p_n = (\log n + c_n)/n$.

(a) Show that, as $n \to \infty$,

$$\mathbb{P}(G(n, p_n) \text{ has no isolated vertices}) \to \begin{cases} 0 & \text{if } c_n \to -\infty, \\ 1 & \text{if } c_n \to \infty. \end{cases}$$

(b) Suppose $c_n \to c \in \mathbb{R}$, compute, with proof, the limit of LHS above as $n \to \infty$, by following the approach in **C3**.

C3. *Poisson limit.* Let X be the number of triangles in $G(n, c/n)$ for some fixed $c > 0$.

(a) For every nonnegative integer k, determine the limit of $\mathbb{E}\binom{X}{k}$ as $n \to \infty$.

(b) Let $Y \sim \text{Binomial}(n, \lambda/n)$ for some fixed $\lambda > 0$. For every nonnegative integer k, determine the limit of $\mathbb{E}\binom{Y}{k}$ as $n \to \infty$, and show that it agrees with the limit in (a) for some $\lambda = \lambda(c)$.

We know that Y converges to the Poisson distribution with mean λ. Also, the Poisson distribution is determined by its moments.

(c) Compute, for fixed every nonnegative integer t, the limit of $\mathbb{P}(X = t)$ as $n \to \infty$. (In particular, this gives the limit probability that $G(n, c/n)$ contains a triangle, i.e., $\lim_{n\to\infty} \mathbb{P}(X > 0)$. This limit increases from 0 to 1 continuously when c ranges from 0 to $+\infty$, thereby showing that the property of containing a triangle has a coarse threshold.)

- <u>No isolated vertices:</u>

$$p = \frac{\log n + c_n}{n}$$

$\mathbb{P}(G(n,p) \text{ has no isolated vertices})$

$\downarrow$

$$\begin{cases} 0 & \text{if } c_n \to -\infty \\ 1 - e^{e^{-c}} & \text{if } c_n \to c \\ 1 & \text{if } c_n \to \infty \end{cases}$$

- <u>Connectivity</u>:

$$p = \frac{\log n + c_n}{n}$$

$$\mathbb{P}(G(n,p) \text{ is connected}) \to \begin{cases} 0 & \text{if } c_n \to -\infty \\ 1 - e^{e^{-c}} & \text{if } c_n \to c \\ 1 & \text{if } c_n \to \infty \end{cases}.$$